Indian Statistical Institute, Bangalore

M.Math I Year, First Semester

Semestral Examination

Algebra I

Time: 3 hours December 05, 2011 Instructor: N.S.N.Sastry

Maximum marks: 100

Note: Answer all questions. Your answers should be complete, precise and to the point.

1. Which of the following $R-$ modules are Noetherian? Justify your answers.

   i) $R[X, Y]/\langle X^2Y + XY^2\rangle$ with R a field;

   ii) $R[[X]]$ with $R = \mathbb{Z}_n$;

   iii) $R[X_1, X_2, X_3, \cdots]$ with $R = \mathbb{Z}$. State the results used for your justifications precisely. [5+5+5=15]

2. a) Define a primary submodule of a module over a commutative ring R (with 1).

   b) Determine the primary submodules of $\mathbb{R}[X]$. [8+6=14]

3. a) Define the exterior algebra associated with a finite dimensional vector space.

   b) If V is a vector space of dimension n over a field k, determine the dimension of the exterior algebra over k. [8+9=17]

4. a) Define the semi-direct product of a group A by a group H.

   b) Use (a) to show that there is only one non-abelian group of order 14.

   [6+14=20]

5. a) Define the tensor product of two $R-$ modules over a commutative ring R. Specify its $R-$ module structure.

   b) i) Find the order of $\mathbb{Z}_m \otimes_{\mathbb{Z}} \mathbb{Z}_n$.

   ii) Show that $\mathbb{C} \otimes (\mathbb{Z} \oplus \mathbb{Z}) \simeq \mathbb{C}^2$, as vector spaces over $\mathbb{C}$. [6+8+8=22]

6. Let F be a field and $f(X) \varepsilon F[X]$. Show that there exists a finite extension field E of F such that $f(X)$ can be written as a product of linear factors with coefficients from E. [12]

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